Problem set 5
Due date: 11th March

(Submit any five)

Exercise 45. Let $Z_1,\dots,Z_n$ be i.i.d $N(0,1)$ and write $\mathbf{Z}$ for the vector with components $Z_1,\dots,Z_n$. Let A be an $m\times n$ matrix and let μ be a vector in $\mathbb{R}^m$. Then the m-dimensional random vector $\mathbf{X}=\mu+A\mathbf{Z}$ is said to have distribution $N_m(\mu,\Sigma)$ where $\Sigma=AA^t$ ('Normal distribution with mean vector μ and covariance matrix Σ').

(1) If $m\le n$ and A has rank m, show that $\mathbf{X}$ has density $(2\pi)^{-\frac{m}{2}}\exp\{-\frac{1}{2}\mathbf{x}^tA^{-1}\mathbf{x}\}$ w.r.t Lebesgue measure on $\mathbb{R}^m$. In particular, note that the distribution depends only on μ and AA^t. (**Note:** If $m>n$ or if $\text{rank}(A)<m$, then satisfy yourself that $\mathbf{X}$ has no density w.r.t Lebesgue measure on $\mathbb{R}^m$ - you do not need to submit this).

(2) Check that $\mathbf{E}[X_i]=\mu_i$ and $\text{Cov}(X_i,X_j)=\Sigma_{i,j}$.

(3) What is the distribution of (i) $(X_1,\dots,X_k)$, for $k\le n$? (ii) $B\mathbf{X}$, where B is a $p\times m$ matrix? (iii) $X_1+\dots+X_m$?

Exercise 46. (1) If X,Y are independent random variables, show that $\text{Cov}(X,Y)=0$.

(2) Give a counterexample to the converse by giving an infinite sequence of random variables $X_1,X_2,\dots$ such that $\text{Cov}(X_i,X_j)=0$ for any $i\neq j$ but such that X_i are not independent.

(3) Suppose $(X_1,\dots,X_m)$ has (joint) normal distribution (see the first question). If $\text{Cov}(X_i,X_j)=0$ for all $i\le k$ and for all $j\ge k+1$, then show that $(X_1,\dots,X_k)$ is independent of $(X_{k+1},\dots,X_m)$.

Exercise 47. Suppose $(X_1,\dots,X_n)$ has density f (w.r.t Lebesgue measure on $\mathbb{R}^2$).

(1) If $f(x_1,\dots,x_n)$ can be written as $\prod_{k=1}^n g_k(x_k)$ for some one-variable functions $g_1,\dots,g_n$. Then show that $X_1,\dots,X_n$ are independent. (Don't assume that g_k is a density!)

(2) If $X_1,\dots,X_n$ are independent, then $f(x_1,\dots,x_n)$ can be written as $\prod_{k=1}^n g_k(x_k)$ for some one-variable densities $g_1,\dots,g_n$.

Exercise 48. Among all $n!$ permutations of $[n]$, pick one at random with uniform probability. Show that the probability that this random permutation has no fixed points is at most $\frac{1}{2}$ for any n.

Exercise 49. Suppose each of $r=\lambda n$ balls are put into n boxes at random (more than one ball can go into a box). If N_n denotes the number of empty boxes, show that for any $\delta>0$, as $n\to\infty$,

$$\mathbf{P}\left(\left|\frac{N_n}{n}-e^{-\lambda}\right|>\delta\right)\to 0$$

Exercise 50. Let X_n be i.i.d random variables such that $\mathbf{E}[|X_1|]<\infty$. Define the random power series $f(z)=\sum_{k=0}^{\infty}X_nz^n$. Show that almost surely, the radius of convergence of f is equal to 1. [**Note:** Recall from Analysis class that the radius of convergence of a power series $\sum c_nz^n$ is given by $(\limsup|c_n|^{\frac{1}{n}})^{-1}$].

Exercise 51. (1) Let X be a real values random variable with finite variance. Show that $f(a):=\mathbf{E}[(X-a)^2]$ is minimized at $a=\mathbf{E}[X]$.

(2) What is the quantity that minimizes $g(a)=\mathbf{E}[|X-a|]$? [**Hint:** First consider X that takes finitely many values with equal probability each].